

# FOUR DIMENSIONAL STATIC AND RELATED CRITICAL SPACES WITH HARMONIC CURVATURE

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**ABSTRACT.** In this article we study any 4-dimensional Riemannian manifold  $(M, g)$  with harmonic curvature which admits a smooth nonzero solution  $f$  to the following equation

$$(0.1) \quad \nabla df = f(Rc - \frac{R}{n-1}g) + xRc + y(R)g.$$

where  $Rc$  is the Ricci tensor of  $g$ ,  $x$  is a constant and  $y(R)$  a function of the scalar curvature  $R$ . We show that a neighborhood of any point in some open dense subset of  $M$  is locally isometric to one of the following five types; (i)  $\mathbb{S}^2(\frac{R}{6}) \times \mathbb{S}^2(\frac{R}{3})$  with  $R > 0$ , (ii)  $\mathbb{H}^2(\frac{R}{6}) \times \mathbb{H}^2(\frac{R}{3})$  with  $R < 0$ , where  $\mathbb{S}^2(k)$  and  $\mathbb{H}^2(k)$  are the two-dimensional Riemannian manifold with constant sectional curvature  $k > 0$  and  $k < 0$ , respectively, (iii) the static spaces in Example 3 below, (iv) conformally flat static spaces described in Kobayashi's [18], and (v) a Ricci flat metric.

We then get a number of Corollaries, including the classification of the following four dimensional spaces with harmonic curvature; static spaces, Miao-Tam critical metrics and  $V$ -static spaces.

The proof is based on the argument from a preceding study of gradient Ricci solitons [17]. Some Codazzi-tensor properties of Ricci tensor, which come from the harmonicity of curvature, are effectively used.

## 1. INTRODUCTION

In this article we consider an  $n$ -dimensional Riemannian manifold  $(M, g)$  with constant scalar curvature  $R$  which admits a smooth nonzero solution  $f$  to the following equation

$$(1.1) \quad \nabla df = f(Rc - \frac{R}{n-1}g) + x \cdot Rc + y(R)g.$$

where  $Rc$  is the Ricci curvature of  $g$ ,  $x$  is a constant and  $y(R)$  a function of  $R$ . There are several well-known classes of spaces which admit such solutions.

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Below we describe them and briefly explain their geometric significances and recent developments.

A *static space* admits by definition a smooth nonzero solution  $f$  to

$$(1.2) \quad \nabla df = f(Rc - \frac{R}{n-1}g).$$

A Riemannian geometric interest of a static space comes from the fact that the scalar curvature functional  $\mathfrak{S}$ , defined on the space  $\mathfrak{M}$  of smooth Riemannian metrics on a closed manifold, is locally surjective at  $g \in \mathfrak{M}$  if there is no nonzero smooth function satisfying (1.2); see Chapter 4 of [3].

This interpretation also holds in local sense. Roughly speaking, if no nonzero smooth function on a compactly contained subdomain  $\Omega$  of a smooth manifold satisfies (1.2) for a Riemannian metric  $g$  on  $\Omega$ , then the scalar curvature functional defined on the space of Riemannian metrics on  $\Omega$  is locally surjective at  $g$  in a natural sense; see Theorem 1 of Corvino [7]. This local viewpoint has been developed to make remarkable progress in Riemannian and Lorentzian geometry [6, 7, 8, 9].

Kobayashi [18] studied a classification of conformally flat static spaces. In his study the list of *complete* ones is made. Moreover, all *local* ones are described for all varying parameter conditions and initial values of the static space equation. Indeed, they belong to the cases I~VI in the section 2 of [18] and the existence of solutions in each case is thoroughly discussed. Lafontaine [14] independently proved a classification of closed conformally flat static space. Qing-Yuan [20] classified complete Bach-flat static spaces which contain compact level hypersurfaces. These spaces are all conformally flat in the case of dimension four.

Next to static spaces we consider a Miao-Tam critical metric [15, 16], which is a compact Riemannian manifold  $(M, g)$  that admits a smooth nonzero solution  $f$ , vanishing at the smooth boundary of  $M$ , to

$$(1.3) \quad \nabla df = f(Rc - \frac{R}{n-1}g) - \frac{g}{n-1}.$$

In [16], Miao-Tam critical metrics are classified when they are Einstein or conformally flat. In [1], Barros, Diógenes and Ribeiro proved that if  $(M^4, g, f)$  is a Bach-flat simply connected, compact Miao-Tam critical metric with boundary isometric to a standard sphere  $S^3$ , then  $(M^4, g)$  is isometric to a geodesic ball in a simply connected space form  $\mathbb{R}^4$ ,  $\mathbb{H}^4$  or  $\mathbb{S}^4$ .

In [8] Corvino, Eichmair and Miao defined a *V-static space* to be a Riemannian manifold  $(M, g)$  which admits a non-trivial solution  $(f, c)$ , for a

constant  $c$ , to the equation

$$(1.4) \quad \nabla df = f(Rc - \frac{R}{n-1}g) - \frac{c}{n-1}g.$$

Note that  $(M, g)$  is a  $V$ -static space if and only if it admits a solution  $f$  to (1.2) or (1.3) on  $M$ , seen by scaling constants. Under a natural assumption, a  $V$ -static metric  $g$  is a critical point of a geometric functional as explained in the theorem 2.3 of [8]. Like static spaces, *local*  $V$ -static spaces are still important; see e.g. theorems 1.1, 1.6 and 2.3 in [8].

Lastly, one may consider Riemannian metrics  $(M, g)$  which admit a non-constant solution  $f$  to

$$(1.5) \quad \nabla df = f(Rc - \frac{R}{n-1}g) + Rc - \frac{R}{n}g.$$

If  $M$  is a closed manifold, then  $g$  is a critical point of the total scalar curvature functional defined on the space of Riemannian metrics with unit volume and with constant scalar curvature on  $M$ . By an abuse of terminology we shall call a metric  $g$  satisfying (1.5) a *critical point metric* even when  $M$  is not closed. There are a number of literatures on this subject, including [3, Section 4.F] and [14, 13, 2, 20].

In this paper we study spaces with harmonic curvature having a non-zero solution to (1.1). It is confined to four dimensional spaces here, but our study may be extendible to higher dimensions. As motivated by the Corvino's local deformation theory of scalar curvature, we study local (i.e. not necessarily complete) classification. We completely characterize non-conformally-flat spaces, so that together with the Kobayashi's work on conformally flat ones we get a full classification as follows.

**Theorem 1.1.** *Let  $(M, g)$  be a four dimensional (not necessarily complete) Riemannian manifold with harmonic curvature, satisfying (1.1) with non-constant  $f$ . Then for each point  $p$  in some open dense subset  $\tilde{M}$  of  $M$ , there exists a neighborhood  $V$  of  $p$  with one of the following properties;*

(i)  *$(V, g)$  is isometric to a domain in  $\mathbb{S}^2(\frac{R}{6}) \times \mathbb{S}^2(\frac{R}{3})$  with  $R > 0$ , where  $\mathbb{S}^2(k)$  is the two-dimensional sphere with constant sectional curvature  $k > 0$ . And  $f = c_1 \cos(\sqrt{\frac{R}{6}}s) - x$  for any constant  $c_1$ , where  $s$  is the distance on  $\mathbb{S}^2(\frac{R}{6})$  from a point. The constant  $R$  equals the scalar curvature of  $g$ . It holds that  $x\frac{R}{3} + y(R) = 0$ .*

(ii)  *$(V, g)$  is isometric to a domain in  $(\mathbb{H}^2(\frac{R}{6}) \times \mathbb{H}^2(\frac{R}{3}), g_{\frac{R}{6}} + g_{\frac{R}{3}})$  with  $R < 0$ , where  $\mathbb{H}^2(k)$  is the hyperbolic plane with constant sectional curvature  $k < 0$  and  $g_k$  is the Riemannian metric of constant curvature  $k$ .  $g_{\frac{R}{6}}$  can be*

written as  $g_{\frac{R}{6}} = ds^2 + p(s)^2 dt^2$  where  $p'' + \frac{R}{6}p = 0$  and then  $f = c_2 p' - x$  for any constant  $c_2$ . It holds that  $x\frac{R}{3} + y(R) = 0$ .

(iii)  $(V, g)$  is isometric to a domain in one of the static spaces in Example 3 of Subsection 2.1.2 below, which is the Riemannian product  $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2 \tilde{g})$  of  $(\mathbb{R}^1, dt^2)$  and some 3-dimensional conformally flat static space  $(W^3, ds^2 + h(s)^2 \tilde{g})$  with zero scalar curvature. And  $f = c \cdot h'(s) - x$ , for any constant  $c$ . It holds that  $R = 0$  and  $y(0) = 0$ .

(iv)  $(V, g)$  is conformally flat. It is one of the metrics whose existence is described in the section 2 of [18];  $g = ds^2 + h(s)^2 g_k$  where  $h$  is a solution of

$$(1.6) \quad h'' + \frac{R}{12}h = ah^{-3}, \quad \text{for a constant } a.$$

For the constant  $k$ , the function  $h$  satisfies

$$(1.7) \quad (h')^2 + ah^{-2} + \frac{R}{12}h^2 = k.$$

And  $f$  is a non-constant solution to the ordinary differential equation for  $f$ ;

$$(1.8) \quad h'f' - fh'' = x(h'' + \frac{R}{3}h) + y(R)h.$$

Conversely, any  $(V, g, f)$  from (i)~(iv) has harmonic curvature and satisfies (1.1).

Theorem 1.1 only considered the case when  $f$  is a nonconstant solution, but the other case of  $f$  being a nonzero constant solution is easier, which is described in Subsection 2.1.1.

Theorem 1.1 yields a number of classification theorems on four dimensional spaces with harmonic curvature as follows. Theorem 8.2 classifies complete spaces satisfying (1.1). Then Theorem 9.1, 10.2 and 11.1 state the classification of *local* static spaces, *V*-static spaces and critical point metrics, respectively. Theorem 9.2 and 11.2 classify complete static spaces and critical point metrics, respectively. Theorem 10.3 gives a characterization of some 4-d Miao-Tam critical metrics with harmonic curvature, which is comparable to the afore-mentioned Bach-flat result [1].

To prove theorem 1.1 we look into the eigenvalues of the Ricci tensor, which is a Codazzi tensor under harmonic curvature condition. This Codazzi tensor encodes some geometric information as investigated by Derdzinsky [10]. In [17] the first-named author has analyzed it in the Ricci soliton setting. We follow the same line of arguments. A crucial part is to show that all Ricci-eigenvalues  $\lambda_i$ ,  $i = 1, \dots, 4$  locally depend on only one function  $s$  such that  $\nabla s = \frac{\nabla f}{|\nabla f|}$ . Then we divide the proof into some cases, depending

on the distinctiveness of  $\lambda_2, \lambda_3, \lambda_4$ . There are two non-trivial cases; when these three are pairwise distinct and when exactly two of them are equal. In the latter case we reduce the analysis to ordinary differential equations, in a way similar to that of [4], and resolve them. And computations on (1.1) using Codazzi tensor property show that the former case does not occur.

This paper is organized as follows. In section 2, we discuss examples and some properties from (1.1) and harmonic curvature. In section 3, we prove that all Ricci-eigenvalues locally depend on only one variable. We study in section 4 the case when the three eigenvalues  $\lambda_2, \lambda_3, \lambda_4$  are pairwise distinct. In section 5 and 6 we analyze the case when exactly two of the three are equal. In section 7 we prove the local classification theorem as Theorem 1. We discuss the classification of complete spaces in section 8. In section 9, 10 and 11 we treat static spaces, Miao-Tam critical &  $V$ -static spaces and critical point metrics respectively.

## 2. EXAMPLES AND PROPERTIES FROM (1.1) AND HARMONIC CURVATURE

We are going to describe some examples of spaces which satisfy (1.1) in Subsection 2.1 and state basic properties of spaces with harmonic curvature satisfying (1.1) in Subsection 2.2.

### 2.1. Examples of spaces satisfying (1.1).

**2.1.1. Spaces with a nonzero constant solution to (1.1).** When  $(M, g)$  has a constant solution  $f = -x$  to (1.1), then  $y(R) + x\frac{R}{n-1} = 0$ . Conversely, any metric with its scalar curvature satisfying  $y(R) + x\frac{R}{n-1} = 0$  admits the constant solution  $f = -x$  to (1.1) because  $\nabla df = f(Rc - \frac{R}{n-1}g) + xRc + y(R)g = (f + x)(Rc - \frac{R}{n-1}g)$ . This proves

**Lemma 2.1.** *An  $n$ -dimensional Riemannian manifold  $(M, g)$  of constant scalar curvature  $R$  admits the constant solution  $f = -x$  if and only if it satisfies  $y(R) + x\frac{R}{n-1} = 0$ .*

If  $(M, g)$  has a constant solution  $f = c_0$ , which does not equal  $-x$ , then  $g$  is an Einstein metric. Conversely, if  $g$  is Einstein, i.e.  $Rc = \frac{R}{n}g$  with  $R \neq 0$ , then any constant  $c_0$  satisfying  $c_0R = (n-1)xR + y(R)n(n-1)$  is a solution to (1.1); but if  $g$  is Ricci-flat, then  $f = c_0$  is a solution exactly when  $y(0) = 0$ .

**2.1.2. Some examples of spaces which satisfy (1.1) with non constant  $f$ .**

#### **Example 1: Einstein spaces satisfying (1.1) with non constant $f$**

Let  $(M, g, f)$  be a 4-dimensional space satisfying (1.1) where  $g$  is an Einstein metric. We shall show that  $g$  has constant sectional curvature. We

may use the argument in the section 1 of Cheeger-Colding [5]. In fact, the relation (1.6) of that paper corresponds to the equation

$$(2.9) \quad \nabla df = \left\{ -\frac{R}{12}f + x\frac{R}{4} + y(R) \right\} g$$

in our Einstein case. One can readily see that their argument still works to get their (1.19); in some neighborhood of any point in  $M$  we can write  $g = ds^2 + (f'(s))^2 \tilde{g}$ , where  $s$  is a variable such that  $\nabla s = \frac{\nabla f}{|\nabla f|}$  and  $\tilde{g}$  is considered as a Riemannian metric on a level surface of  $f$ .

As  $g$  is Einstein, so is  $\tilde{g}$  from Derdziński's Lemma 4 in [10]. As  $\tilde{g}$  is 3-dimensional, it has constant sectional curvature, say  $k$ . And  $f$  satisfies  $f'' = -\frac{R}{12}f + x\frac{R}{4} + y(R)$ , by feeding  $(\frac{\partial}{\partial s}, \frac{\partial}{\partial s})$  to (2.9).

Since  $g$  is Einstein, we can readily see that our warped product metric  $g$  has constant sectional curvature. In particular, a 4-d complete positive Einstein space satisfying (1.1) with non constant  $f$  is a round sphere; cf. [19, 21].

**Example 2** Assume that  $x\frac{R}{3} + y(R) = 0$ . Then (1.1) reduces to  $\nabla df = (f+x)(Rc - \frac{R}{n-1}g)$ . This is the static space equation for  $g$  and  $F = f+x$ . We recall one example from [14]. On the round sphere  $\mathbb{S}^2(1)$  of sectional curvature 1, we consider the local coordinates  $(s, t) \in (0, \pi) \times \mathbb{S}^1$  so that the round metric is written  $ds^2 + \sin^2(s) dt^2$ . Let  $f(s) = c_1 \cos s - x$  for any constant  $c_1$ . Then the product metric of  $\mathbb{S}^2(1) \times \mathbb{S}^2(2)$  with  $f$  satisfies (1.1).

This example is not Einstein nor conformally flat.

**Example 3** Here we shall describe some 4-d non-conformally-flat static space  $g_W + dt^2$ . We first recall some spaces among Kobayashi's warped product static spaces [18] on  $I \times N(k)$  with the metric  $g = ds^2 + r(s)^2 \bar{g}$ , where  $I$  is an interval and  $(\bar{g}, N(k))$  is a  $(n-1)$ -dimensional Riemannian manifold of constant sectional curvature  $k$ . And  $f = cr'$  for a nonzero constant  $c$ .

In order for  $g$  to be a static space, the next equation needs to be satisfied; for a constant  $\alpha$

$$(2.10) \quad r'' + \frac{R}{n(n-1)}r = \alpha r^{1-n},$$

along with an integrability condition; for a constant  $k$ ,

$$(2.11) \quad (r')^2 + \frac{2\alpha}{n-2}r^{2-n} + \frac{R}{n(n-1)}r^2 = k.$$

Existence of solutions depends on the values of  $\alpha, R, k$ . Here we consider only when  $R = 0$ . Then there are three cases:

- (i)  $R = 0, \alpha > 0$ ,      (ii)  $R = 0, \alpha < 0$ ,      (iii)  $R = 0, \alpha = 0$ .

The above (i), (ii) and (iii) correspond respectively to the case IV.1, III.1 and II in Section 2 of [18]. The solutions for these cases are discussed in Proposition 2.5, Example 5 and Proposition 2.4 in that paper. In particular, if  $R = 0, \alpha > 0$  (then  $k > 0$ ) and  $n = 3$ , we get the warped product metric on  $\mathbb{R}^1 \times \mathbb{S}^2(1)$  which contains the space section of the Schwarzschild space-time. Next, if  $R = 0, \alpha < 0$ , then there is an incomplete metric on  $I \times N(k)$ . If  $R = 0, \alpha = 0$ , then  $g$  is readily a flat metric.

Let  $(W^3, g_W, f)$  be one of the 3-dimensional static spaces  $(g, f)$  in the above paragraph. We now consider the 4-dimensional product metric  $g_W + dt^2$  on  $W^3 \times \mathbb{R}^1$ . One can check that  $(W^3 \times \mathbb{R}^1, g_W + dt^2, f \circ \text{pr}_1)$  is a static space, where  $\text{pr}_1$  is the projection of  $W^3 \times \mathbb{R}^1$  onto the first factor. The metric  $g_W + dt^2$  is not conformally flat and has three distinct *Ricci* eigenvalues.

**2.2. Spaces with harmonic curvature.** We begin with a basic formula;

**Lemma 2.2.** *For a 4-dimensional manifold  $(M^4, g, f)$  with harmonic curvature satisfying (1.1), it holds that*

$$\begin{aligned} -R(X, Y, Z, \nabla f) &= -R(X, Z)g(\nabla f, Y) + R(Y, Z)g(\nabla f, X) \\ &\quad - \frac{R}{3}\{g(\nabla f, X)g(Y, Z) - g(\nabla f, Y)g(X, Z)\}. \end{aligned}$$

*Proof.* By Ricci identity,  $\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = -R_{ijkl} \nabla_l f$ . The equation (1.1) gives

$$\begin{aligned} -R_{ijkl} \nabla_l f &= \nabla_i \{f(R_{jk} - \tfrac{1}{3}Rg_{jk}) + xR_{jk} + y(R)g_{jk}\} \\ &\quad - \nabla_j \{f(R_{ik} - \tfrac{1}{3}Rg_{ik}) + xR_{ik} + y(R)g_{ik}\} \\ &= f_i(R_{jk} - \tfrac{1}{3}Rg_{jk}) - f_j(R_{ik} - \tfrac{1}{3}Rg_{ik}), \end{aligned}$$

which yields the lemma.  $\square$

A Riemannian manifold with harmonic curvature is real analytic in harmonic coordinates [11]. The equation (1.1) then tells that  $f$  is real analytic in harmonic coordinates.

One may mimic arguments in [4] and get the next lemma.

**Lemma 2.3.** *Let  $(M^n, g, f)$  have harmonic curvature, satisfying (1.1) with nonconstant  $f$ . Let  $c$  be a regular value of  $f$  and  $\Sigma_c = \{x | f(x) = c\}$  be the level surface of  $f$ . Then the followings hold;*

- (i) *Where  $\nabla f \neq 0$ ,  $E_1 := \frac{\nabla f}{|\nabla f|}$  is an eigenvector field of  $Rc$ .*
- (ii)  *$|\nabla f|$  is constant on a connected component of  $\Sigma_c$ .*
- (iii) *There is a function  $s$  locally defined with  $s(x) = \int \frac{df}{|\nabla f|}$ , so that*  

$$ds = \frac{df}{|\nabla f|} \text{ and } E_1 = \nabla s.$$
- (iv)  *$R(E_1, E_1)$  is constant on a connected component of  $\Sigma_c$ .*

(v) Near a point in  $\Sigma_c$ , the metric  $g$  can be written as

$$g = ds^2 + \sum_{i,j>1} g_{ij}(s, x_2, \dots, x_n) dx_i \otimes dx_j, \text{ where } x_2, \dots, x_n \text{ is a local coordinates system on } \Sigma_c.$$

(vi)  $\nabla_{E_1} E_1 = 0$ .

*Proof.* In Lemma 2.2, put  $Y = Z = \nabla f$  and  $X \perp \nabla f$  to get

$$0 = -R(X, \nabla f, \nabla f, \nabla f) = -R(X, \nabla f)g(\nabla f, \nabla f). \text{ So, } R(X, \nabla f) = 0. \\ \text{Hence } E_1 = \frac{\nabla f}{|\nabla f|} \text{ is an eigenvector of } Rc. \text{ Also, } \frac{1}{2} \nabla_X |\nabla f|^2 = \langle \nabla_X \nabla f, \nabla f \rangle = \\ fR(\nabla f, X) = 0 \text{ for } X \perp \nabla f. \text{ We proved (ii). Next } d\left(\frac{df}{|\nabla f|}\right) = -\frac{1}{2|\nabla f|^{\frac{3}{2}}} d|\nabla f|^2 \wedge$$

$df = 0$  as  $\nabla_X (|\nabla f|^2) = 0$ . So, (iii) is proved. It holds that  $(\nabla_{E_1} E_1)f = 0$  because, setting  $\nabla_{E_1} E_1(f) = \sum_i \langle \nabla_{E_1} E_1, E_i \rangle E_i(f) = \langle \nabla_{E_1} E_1, E_1 \rangle E_1(f)$ , then  $2\langle \nabla_{E_1} E_1, E_1 \rangle = E_1 \langle E_1, E_1 \rangle = 0$ . We compute  $\nabla_g df(E_1, E_1) = f(Ric_g - \frac{1}{n-1} R_g g)(E_1, E_1) + xR(E_1, E_1) + y(R)$ . Then  $E_1 E_1 f - (\nabla_{E_1} E_1)f = E_1 E_1 f = f(R(E_1, E_1) - \frac{1}{n-1} R) + xR(E_1, E_1) + y(R)$ . Since  $f$  is not zero on an open subset, so  $R(E_1, E_1)$  is constant on a connected component of  $\Sigma_c$ . As  $\nabla f$  and the level surfaces of  $f$  are perpendicular, one gets (v).

One uses (v) to compute Christoffel symbols and gets (vi).  $\square$

The Ricci tensor  $r$  of a Riemannian metric with harmonic curvature is a Codazzi tensor, written in local coordinates as  $\nabla_k R_{ij} = \nabla_i R_{kj}$ . Here  $Rc$  or  $r$  denotes the Ricci tensor, but its components in vector frames shall be written as  $R_{ij}$ . Following Derdziński [10], for a point  $x$  in  $M$ , let  $E_r(x)$  be the number of distinct eigenvalues of  $r_x$ , and set  $M_r = \{x \in M \mid E_r \text{ is constant in a neighborhood of } x\}$ , so that  $M_r$  is an open dense subset of  $M$ . Then we have;

**Lemma 2.4.** *For a Riemannian metric  $g$  of dimension  $n \geq 4$  with harmonic curvature, consider orthonormal vector fields  $E_i, i = 1, \dots, n$  such that  $R(E_i, \cdot) = \lambda_i g(E_i, \cdot)$ . Then the followings hold in each connected component of  $M_r$ ;*

$$(i) (\lambda_j - \lambda_k) \langle \nabla_{E_i} E_j, E_k \rangle + E_i \{R(E_j, E_k)\} = (\lambda_i - \lambda_k) \langle \nabla_{E_j} E_i, E_k \rangle + E_j \{R(E_k, E_i)\}, \\ \text{for any } i, j, k = 1, \dots, n.$$

$$(ii) \text{ If } k \neq i \text{ and } k \neq j, \quad (\lambda_j - \lambda_k) \langle \nabla_{E_i} E_j, E_k \rangle = (\lambda_i - \lambda_k) \langle \nabla_{E_j} E_i, E_k \rangle.$$

(iii) *Given distinct eigenfunctions  $\lambda, \mu$  of  $A$  and local vector fields  $v, u$  such that  $Av = \lambda v, Au = \mu u$  with  $|u| = 1$ , it holds that*

$$v(\mu) = (\mu - \lambda) \langle \nabla_u u, v \rangle.$$

(iv) *For each eigenfunction  $\lambda$ , the  $\lambda$ -eigenspace distribution is integrable and its leaves are totally umbilic submanifolds of  $M$ .*

*Proof.* The statement (i) was proved in [17]. And (ii) and (iii) follow from (i). (iii) and (iv) are from the section 2 of [10].  $\square$



Given  $(M^n, g, f)$  with harmonic curvature satisfying (1.1),  $f$  is real analytic in harmonic coordinates, so  $\{\nabla f \neq 0\}$  is open and dense in  $M$ . Lemma 2.3 gives that for any point  $p$  in the open dense subset  $M_r \cap \{\nabla f \neq 0\}$  of  $M^n$ , there is a neighborhood  $U$  of  $p$  where there exists an orthonormal Ricci-eigen vector fields  $E_i$ ,  $i = 1, \dots, n$  such that

- (i)  $E_1 = \frac{\nabla f}{|\nabla f|}$ ,
- (ii) for  $i > 1$ ,  $E_i$  is tangent to smooth level hypersurfaces of  $f$ .

These local orthonormal Ricci-eigen vector fields  $\{E_i\}$  shall be called an *adapted frame field* of  $(M, g, f)$ .

### 3. CONSTANCY OF $\lambda_i$ ON LEVEL HYPERSURFACES OF $f$

For an adapted frame field of  $(M^n, g, f)$  with harmonic curvature satisfying (1.1), we set  $\zeta_i := -\langle \nabla_{E_i} E_i, E_1 \rangle = \langle E_i, \nabla_{E_i} E_1 \rangle$ , for  $i > 1$ . Then  $\nabla_{E_i} E_1 = \nabla_{E_i} \left( \frac{\nabla f}{|\nabla f|} \right) = \frac{\nabla_{E_i} \nabla f}{|\nabla f|} = \frac{fR(E_i, \cdot) - f \frac{R}{n-1} g(E_i, \cdot) + xR(E_i, \cdot) + y(R)g(E_i, \cdot)}{|\nabla f|}$ . So we may write;

$$(3.12) \quad \nabla_{E_i} E_1 = \zeta_i E_i \quad \text{where } \zeta_i = \frac{(f+x)R(E_i, E_i) - \frac{R}{n-1}f + y(R)}{|\nabla f|}.$$

Due to Lemma 2.3, in a neighborhood of a point  $p \in M_r \cap \{\nabla f \neq 0\}$ ,  $f$  may be considered as functions of the variable  $s$  only, and we write the derivative in  $s$  by a prime:  $f' = \frac{df}{ds}$ .

**Lemma 3.1.** *Let  $(M, g, f)$  be a 4-dimensional space with harmonic curvature, satisfying (1.1) with nonconstant  $f$ . The Ricci eigen-functions  $\lambda_i$  associated to an adapted frame field  $E_i$  are constant on a connected component of a regular level hypersurface  $\Sigma_c$  of  $f$ , and so depend on the local variable  $s$  only. And  $\zeta_i$ ,  $i = 2, 3, 4$ , in (3.12) also depend on  $s$  only. In particular, we have  $E_i(\lambda_j) = E_i(\zeta_k) = 0$  for  $i, k > 1$  and any  $j$ .*

*Proof.* We use  $f_{ij} = f(R_{ij} - \frac{1}{3}Rg_{ij}) + xR_{ij} + y(R)g_{ij}$  to compute;

$$\begin{aligned} \sum_{j=1}^4 \frac{1}{2} \nabla_{E_j} \nabla_{E_j} (|\nabla f|^2) &= \sum_{i,j} \frac{1}{2} \nabla_{E_j} \nabla_{E_j} (f_i f_i) \\ &= \sum_{i,j} \nabla_{E_j} (f_i f_{ij}) = \sum_{i,j} \nabla_{E_j} \left\{ f f_i (R_{ij} - \frac{R}{3}g_{ij}) + x f_i R_{ij} + y(R) f_i g_{ij} \right\} \\ &= \sum_{i,j} f_j f_i (R_{ij} - \frac{R}{3}g_{ij}) + f f_{ij} (R_{ij} - \frac{R}{3}g_{ij}) + x f_{ij} R_{ij} + y(R) f_{ij} g_{ij} \\ &= (R_{11} - \frac{R}{3}) |\nabla f|^2 + \sum_{i,j} (f+x)^2 R_{ij} R_{ij} - \frac{2R^2}{9} f^2 - \frac{2xR^2}{3} f \\ &\quad + (2x - \frac{2f}{3}) y(R) R + 4y(R)^2. \end{aligned}$$

Meanwhile,

$$\begin{aligned} \sum_{j=1}^4 \nabla_{E_j} \nabla_{E_j} (|\nabla f|^2) &= \sum_{j=1}^4 E_j E_j (|\nabla f|^2) - (\nabla_{E_j} E_j) (|\nabla f|^2) \\ &= (|\nabla f|^2)'' + \sum_{j=2}^4 \zeta_j (|\nabla f|^2)'. \end{aligned}$$

Since  $R$  and  $\lambda_1 = R_{11}$  depend on  $s$  only by Lemma 2.3, the function  $\sum_{j=2}^4 \zeta_j$  depends only on  $s$  by (3.12). We compare the above two expressions

of  $\sum_{j=1}^4 \nabla_{E_j} \nabla_{E_j} (|\nabla f|^2)$  to see that  $\sum_{i,j} (f+x)^2 R_{ij} R_{ij}$  depends only on  $s$ . As  $f$  is nonconstant real analytic,  $\sum_{i,j} R_{ij} R_{ij}$  depends only on  $s$ .

Below we drop summation symbols.

$$\begin{aligned}
\nabla_k (f_i f_{ij} R_{jk}) &= \nabla_k \{ f_i R_{jk} (f(R_{ij} - \frac{1}{3} R g_{ij}) + x R_{ij} + y(R) g_{ij}) \} \\
&= \nabla_k \{ f_i ((f+x) R_{ij} R_{jk} - \frac{f}{3} R R_{ik} + y(R) R_{ik}) \} \\
&= f_{ik} \{ (f+x) R_{ij} R_{jk} - \frac{f}{3} R R_{ik} + y(R) R_{ik} \} \\
&\quad + f_i (f_k R_{ij} R_{jk} + (f+x) R_{jk} \nabla_k R_{ij} - \frac{f_k}{3} R R_{ik}) \\
&= \{ (f+x) R_{ik} - \frac{f}{3} R g_{ik} + y(R) g_{ik} \} \{ (f+x) R_{ij} R_{jk} - \frac{f}{3} R R_{ik} + y(R) R_{ik} \} \\
&\quad + f_i f_k R_{ij} R_{jk} + (f+x) f_i R_{jk} \nabla_k R_{ij} - \frac{f_i f_k}{3} R R_{ik} \\
&= (f+x)^2 R_{ik} R_{ij} R_{jk} + (f+x) f_i R_{jk} \nabla_k R_{ij} + L(s),
\end{aligned}$$

where  $L(s)$  is a function of  $s$  only.

Using  $\nabla_k R_{ij} = \nabla_i R_{jk}$ , we get

$$(3.13) \quad \nabla_k (f_i f_{ij} R_{jk}) = (f+x)^2 R_{ik} R_{ij} R_{jk} + \frac{(f+x)}{2} f_i \nabla_i (R_{jk} R_{jk}) + L(s).$$

All terms except  $(f+x)^2 R_{ij} R_{jk} R_{ik}$  in the right hand side of (3.13) depend on  $s$  only. From the constancy of  $R$  and (3.12) we also get

$$\begin{aligned}
2\nabla_k (f_i f_{ij} R_{jk}) &= \nabla_k (2f_i f_{ij}) \cdot R_{jk} = \nabla_k \nabla_j (f_i f_i) \cdot R_{jk} \\
&= \sum_{j,k,i} E_k E_j (f_i f_i) \cdot R_{jk} - (\nabla_{E_k} E_j) (f_i f_i) \cdot R_{jk} \\
&= \sum_{j,i} E_j E_j (f_i f_i) \cdot R_{jj} - (\nabla_{E_j} E_j) (f_i f_i) \cdot R_{jj} \\
&= \sum_i E_1 E_1 (f_i f_i) \cdot R_{11} + \sum_{j=2}^4 \zeta_j E_1 (f_i f_i) \cdot R_{jj} \\
(3.14) \quad &= (|\nabla f|^2)'' \cdot R_{11} + \sum_{j=2}^4 \frac{(f+x) R_{jj} R_{jj} - \frac{R}{3} f R_{jj} + y(R) R_{jj}}{|\nabla f|} E_1 (|\nabla f|^2)
\end{aligned}$$

which depends only on  $s$ .

So, we compare (3.13) with (3.14) to see that  $R_{ij} R_{jk} R_{ik}$  depends only on  $s$ . Now  $\lambda_1$  and  $\sum_{i=1}^4 (\lambda_i)^k$ ,  $k = 1, \dots, 3$ , depend only on  $s$ . This implies that each  $\lambda_i$ ,  $i = 1, \dots, 4$ , depends only on  $s$ . By (3.12),  $\zeta_i$ ,  $i = 2, 3, 4$  depend on  $s$  only.

□

4. 4-DIMENSIONAL SPACE WITH DISTINCT  $\lambda_2, \lambda_3, \lambda_4$ 

Let  $(M, g, f)$  be a four dimensional Riemannian manifold with harmonic curvature satisfying (1.1). For an adapted frame field  $\{E_j\}$  with its eigenfunction  $\lambda_j$  in an open subset of  $M_r \cap \{\nabla f \neq 0\}$ , we may only consider three cases depending on the distinctiveness of  $\lambda_2, \lambda_3, \lambda_4$ ; the first case is when  $\lambda_i, i = 2, 3, 4$  are all equal (on an open subset), and the second is when exactly two of the three are equal. And the last is when the three  $\lambda_i, i = 2, 3, 4$ , are mutually different. In this section we shall study the last case. We set  $\Gamma_{ij}^k := \langle \nabla_{E_i} E_j, E_k \rangle$ .

**Lemma 4.1.** *Let  $(M, g, f)$  be a four dimensional Riemannian manifold with harmonic curvature satisfying (1.1) with nonconstant  $f$ . Suppose that for an adapted frame fields  $E_j, j = 1, 2, 3, 4$ , in an open subset  $W$  of  $M_r \cap \{\nabla f \neq 0\}$ , the eigenfunctions  $\lambda_2, \lambda_3, \lambda_4$  are distinct from each other. Then the following holds in  $W$ ;*

$$\begin{aligned} \text{For distinct } i, j > 1, \quad R_{1ii1} &= -\zeta'_i - \zeta_i^2, \quad R_{1ij1} = 0. \\ R_{11} &= -\zeta'_2 - \zeta_2^2 - \zeta'_3 - \zeta_3^2 - \zeta'_4 - \zeta_4^2. \\ R_{22} &= -\zeta'_2 - \zeta_2^2 - \zeta_2 \zeta_3 - \zeta_2 \zeta_4 - 2\Gamma_{34}^2 \Gamma_{43}^2. \\ R_{33} &= -\zeta'_3 - \zeta_3^2 - \zeta_3 \zeta_2 - \zeta_3 \zeta_4 + 2\frac{(\zeta_2 - \zeta_4)}{\zeta_3 - \zeta_4} \Gamma_{34}^2 \Gamma_{43}^2. \\ R_{44} &= -\zeta'_4 - \zeta_4^2 - \zeta_4 \zeta_2 - \zeta_4 \zeta_3 + 2\frac{(\zeta_2 - \zeta_3)}{\zeta_4 - \zeta_3} \Gamma_{34}^2 \Gamma_{43}^2. \\ R_{1j1j} &= R_{jj} - \frac{R}{3}. \end{aligned}$$

*Proof.*  $\nabla_{E_1} E_1 = 0$  from Lemma 2.3 (vi) and  $\nabla_{E_i} E_1 = \zeta_i E_i$  from (3.12). Let  $i, j > 1$  be distinct. From Lemma 2.4 (iii) and Lemma 3.1,  $\langle \nabla_{E_i} E_i, E_j \rangle = 0$ . And  $\langle \nabla_{E_i} E_i, E_1 \rangle = -\langle E_i, \nabla_{E_i} E_1 \rangle = -\zeta_i$ . So, we get  $\nabla_{E_i} E_i = -\zeta_i E_1$ . Now,  $\langle \nabla_{E_i} E_j, E_i \rangle = -\langle \nabla_{E_i} E_i, E_j \rangle = 0$ ,  $\langle \nabla_{E_i} E_j, E_j \rangle = 0$ . And  $\langle \nabla_{E_i} E_j, E_1 \rangle = -\langle \nabla_{E_i} E_1, E_j \rangle = 0$ . So,  $\nabla_{E_i} E_j = \sum_{k \neq 1, i, j} \Gamma_{ij}^k E_k$ . Clearly  $\Gamma_{ij}^k = -\Gamma_{ik}^j$ . From Lemma 2.4 (ii),  $(\lambda_i - \lambda_j) \langle \nabla_{E_1} E_i, E_j \rangle = (\lambda_1 - \lambda_j) \langle \nabla_{E_i} E_1, E_j \rangle$ . As  $\langle \nabla_{E_i} E_1, E_j \rangle = 0$ ,  $\langle \nabla_{E_1} E_i, E_j \rangle = 0$ . This gives  $\nabla_{E_1} E_i = 0$ . Summarizing, we have got;

For  $i, j > 1, i \neq j$ ,

$$\begin{aligned} \nabla_{E_1} E_1 &= 0, \quad \nabla_{E_i} E_1 = \zeta_i E_i, \quad \nabla_{E_i} E_i = -\zeta_i E_1, \quad \nabla_{E_1} E_i = 0. \\ \nabla_{E_i} E_j &= \sum_{k \neq 1, i, j} \Gamma_{ij}^k E_k. \end{aligned}$$

One uses Lemma 3.1 in computing curvature components. We get  $R_{1ii1} = -\zeta'_i - \zeta_i^2$ , and for distinct  $i, j, k > 1$ ,  $R_{jii j} = -\zeta_j \zeta_i - \Gamma_{ji}^k \Gamma_{ik}^j - \Gamma_{ji}^k \Gamma_{ki}^j + \Gamma_{ij}^k \Gamma_{ki}^j$  and  $R_{kij k} = E_k(\Gamma_{ij}^k)$ . And  $R_{1ij1} = 0$ .

From Lemma 2.4, for distinct  $i, j, k > 1$ , we have

$$(4.15) \quad (\zeta_j - \zeta_k) \Gamma_{ij}^k = (\zeta_i - \zeta_k) \Gamma_{ji}^k,$$

which helps to express  $R_{ii}$ . Lemma 2.2 gives  $-R(E_1, E_j, E_j, \nabla f) = (R_{jj} - \frac{R}{3})g(\nabla f, E_1)$  for  $j > 1$ . From this we get

$$(4.16) \quad R_{1j1j} = R_{jj} - \frac{R}{3}.$$

□

From the proof of the above Lemma, we may write

$$(4.17) \quad [E_2, E_3] = \alpha E_4, \quad [E_3, E_4] = \beta E_2, \quad [E_4, E_2] = \gamma E_3.$$

From Jacobi identity  $[[E_1, E_2], E_3] + [[E_2, E_3], E_1] + [[E_3, E_1], E_2] = 0$ , we have

$$(4.18) \quad E_1(\alpha) = \alpha(\zeta_4 - \zeta_2 - \zeta_3).$$

And (4.15) gives;

$$(4.19) \quad \beta = \frac{(\zeta_3 - \zeta_4)^2}{(\zeta_2 - \zeta_3)^2} \alpha, \quad \gamma = \frac{(\zeta_2 - \zeta_4)^2}{(\zeta_2 - \zeta_3)^2} \alpha.$$

We set  $a := \zeta_2$ ,  $b := \zeta_3$  and  $c := \zeta_4$ . From (1.1),  $\zeta_i f' = f(R_{ii} - \frac{R}{3}) + xR_{ii} + y(R)$  for  $i > 1$ . With this and Lemma 4.1,

$$(a - b) \frac{f'}{f} = (1 + \frac{x}{f})(R_{22} - R_{33}) = (1 + \frac{x}{f})\{(b - a)c - 2\{1 + \frac{(a-c)}{b-c}\}\Gamma_{34}^2 \Gamma_{43}^2\}.$$

So,

$$(4.20) \quad -\frac{f'}{f} = (1 + \frac{x}{f})\{c + 2\frac{(a+b-2c)}{(a-b)(b-c)}\Gamma_{34}^2 \Gamma_{43}^2\}.$$

Similarly,  $(a - c) \frac{f'}{f} = (1 + \frac{x}{f})(R_{22} - R_{44}) = (1 + \frac{x}{f})\{(c - a)b - 2\{1 + \frac{(a-b)}{c-b}\}\Gamma_{34}^2 \Gamma_{43}^2\}$ . So,

$$(4.21) \quad -\frac{f'}{f} = (1 + \frac{x}{f})\{b + 2\frac{(a+c-2b)}{(a-c)(c-b)}\Gamma_{34}^2 \Gamma_{43}^2\}.$$

From (4.20) and (4.21), we get

$$(4.22) \quad 4\Gamma_{34}^2 \Gamma_{43}^2 = \frac{(a-b)(a-c)(b-c)^2}{(a^2 + b^2 + c^2 - ab - bc - ac)},$$

$$(4.23) \quad -\frac{f'}{f} = (1 + \frac{x}{f}) \frac{a^2b + a^2c + ab^2 + ac^2 + b^2c + c^2b - 6abc}{2(a^2 + b^2 + c^2 - ab - bc - ac)}.$$

The formula (4.16) gives  $R_{1212} - R_{1313} = R_{22} - R_{33}$ , which reduces to

$$\begin{aligned} 2(a' - b') &= -2(a^2 - b^2) + bc - ac + \frac{(a-b)(b-c)(c-a)(a+b-2c)}{2(a^2+b^2+c^2-ab-bc-ac)} \\ (4.24) \quad &= -2(a^2 - b^2) + \frac{(a-b)}{2P}A, \end{aligned}$$

where we set  $P := a^2 + b^2 + c^2 - ab - bc - ac$  and  $A := 6abc - a^2b - ab^2 - a^2c - ac^2 - b^2c - bc^2$ . By symmetry, we get

$$(4.25) \quad \zeta'_i - \zeta'_j = -(\zeta_i^2 - \zeta_j^2) + \frac{(\zeta_i - \zeta_j)}{4P}A, \quad \text{for } i, j \in \{2, 3, 4\}.$$

The formula (4.25) looks different from the corresponding one in the soliton's study in [17];  $\zeta'_i - \zeta'_j = -(\zeta_i^2 - \zeta_j^2)$ . But surprisingly the next proposition still works through in resolving (1.1); refer to the proposition 3.4 in [17]. Here the formula (4.23) is crucial.

**Proposition 4.2.** *Let  $(M, g, f)$  be a four dimensional Riemannian manifold with harmonic curvature, satisfying (1.1) with nonconstant  $f$ . For any adapted frame field  $E_j$ ,  $j = 1, 2, 3, 4$ , in an open dense subset  $M_r \cap \{\nabla f \neq 0\}$  of  $M$ , the three eigenfunctions  $\lambda_2, \lambda_3, \lambda_4$  cannot be pairwise distinct, i.e. at least two of the three coincide.*

*Proof.* Suppose that  $\lambda_2, \lambda_3, \lambda_4$  are pairwise distinct. We shall prove then that  $f$  should be a constant, a contradiction to the hypothesis.

From (4.22) and (4.15),

$$(\alpha - \gamma + \beta)^2 = 4(\Gamma_{34}^2)^2 = 4\Gamma_{34}^2\Gamma_{43}^2 \frac{(a-b)}{(a-c)} = \frac{(a-b)^2(b-c)^2}{(a^2+b^2+c^2-ab-bc-ac)}.$$

From (4.19),

$$(\alpha - \gamma + \beta)^2 = \alpha^2 \left\{ 1 - \frac{(a-c)^2}{(a-b)^2} + \frac{(b-c)^2}{(a-b)^2} \right\}^2 = \frac{4\alpha^2(b-c)^2}{(a-b)^2}.$$

So,  $\alpha^2 = \frac{(a-b)^4}{4P}$ . Since  $a, b, c$  are all functions of  $s$  only, so is  $\alpha$ . We compute from (4.25)

$$\begin{aligned} (a-b)(a' - b') + (a-c)(a' - c') + (b-c)(b' - c') \\ &= -(a-b)(a^2 - b^2) - (a-c)(a^2 - c^2) - (b-c)(b^2 - c^2) \\ &\quad + \frac{A}{4P} \{ (a-b)^2 + (a-c)^2 + (b-c)^2 \} \\ &= -2(a^3 + b^3 + c^3) + a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + \frac{A}{2} \\ (4.26) \quad &= -2(a^3 + b^3 + c^3 - 3abc) - \frac{1}{2}A \end{aligned}$$

Differentiating  $\alpha^2 = \frac{(a-b)^4}{4P}$  in  $s$  and using (4.25) and (4.26),

$$\begin{aligned}
2\alpha\alpha' &= \frac{(a-b)^3(a'-b')}{P} - \frac{(a-b)^4(2aa'+2bb'+2cc'-ab'-ba'-ac'-ca'-cb'-bc')}{4P^2} \\
&= \frac{-(a-b)^3(a^2-b^2)}{P} + \frac{(a-b)^4}{4P^2}A - \frac{(a-b)^4\{(a-b)(a'-b')+(a-c)(a'-c')+(b-c)(b'-c')\}}{4P^2} \\
&= -\frac{(a-b)^4(a+b)}{P} + \frac{(a-b)^4}{4P^2}A + \frac{(a-b)^4\{2(a^3+b^3+c^3-3abc)\}}{4P^2} + \frac{(a-b)^4\{\frac{1}{2}A\}}{4P^2} \\
&= -\frac{(a-b)^4(a+b-c)}{P} + \frac{3(a-b)^4}{8P^2}A
\end{aligned}$$

Meanwhile, from (4.18) and  $\alpha^2 = \frac{(a-b)^4}{4P}$ ,

$$2\alpha\alpha' = 2\alpha^2(c-a-b) = -\frac{(a-b)^4}{2P}(a+b-c).$$

Equating these two expressions for  $2\alpha\alpha'$ , we get  $A = 0$ . From (4.23),  $f' = 0$ .  $\square$

#### 5. 4-DIMENSIONAL SPACE WITH $\lambda_2 \neq \lambda_3 = \lambda_4$

In this section we study when exactly two of  $\lambda_2, \lambda_3, \lambda_4$  are equal. We may well assume that  $\lambda_2 \neq \lambda_3 = \lambda_4$ . We use (3.12), Lemma 2.4 and Lemma 3.1 to compute  $\nabla_{E_i}E_j$ 's and get the next lemma.

**Lemma 5.1.** *Let  $(M, g, f)$  be a four dimensional Riemannian manifold with harmonic curvature satisfying (1.1) with nonconstant  $f$ . Suppose that  $\lambda_2 \neq \lambda_3 = \lambda_4$  for an adapted frame fields  $E_j$ ,  $j = 1, 2, 3, 4$ , on an open subset  $U$  of  $M_r \cap \{\nabla f \neq 0\}$ . Then we have;*

$$[E_1, E_2] = -\zeta_2 E_2 \text{ and } [E_3, E_4] = \beta_3 E_3 + \beta_4 E_4, \text{ for some functions } \beta_3, \beta_4.$$

*In particular, the distribution spanned by  $E_1$  and  $E_2$  is integrable. So is that spanned by  $E_3$  and  $E_4$ .*

*Proof.* From Lemma 2.4 (ii) and (3.12),  $(\lambda_2 - \lambda_i)\langle \nabla_{E_1}E_2, E_i \rangle = (\lambda_1 - \lambda_i)\langle \nabla_{E_2}E_1, E_i \rangle = (\lambda_1 - \lambda_i)\langle \zeta_2 E_2, E_i \rangle = 0$ , for  $i = 3, 4$ . This gives  $\nabla_{E_1}E_2 = 0$ , and so  $[E_1, E_2] = -\zeta_2 E_2$ .

From Lemma 2.4 (ii),  $(\lambda_2 - \lambda_4)\langle \nabla_{E_3}E_2, E_4 \rangle = (\lambda_3 - \lambda_4)\langle \nabla_{E_2}E_3, E_4 \rangle = 0$ . So,  $\langle \nabla_{E_3}E_2, E_4 \rangle = -\langle E_2, \nabla_{E_3}E_4 \rangle = 0$ . This and (3.12) yields  $\nabla_{E_3}E_4 = \beta_3 E_3$ , for some function  $\beta_3$ . Similarly,  $\nabla_{E_4}E_3 = -\beta_4 E_4$  for some function  $\beta_4$ . Then  $[E_3, E_4] = \beta_3 E_3 + \beta_4 E_4$ .  $\square$

We express the metric  $g$  in some coordinates as in the following lemma.

**Lemma 5.2.** *Under the same hypothesis as Lemma 5.1,*

*for each point  $p_0$  in  $U$ , there exists a neighborhood  $V$  of  $p_0$  in  $U$  with coordinates  $(s, t, x_3, x_4)$  such that  $\nabla s = \frac{\nabla f}{|\nabla f|}$  and  $g$  can be written on  $V$  as*

$$(5.27) \quad g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g},$$

where  $p := p(s)$  and  $h := h(s)$  are smooth functions of  $s$  and  $\tilde{g}$  is (a pull-back of) a Riemannian metric of constant curvature, say  $k$ , on a 2-dimensional domain with  $x_3, x_4$  coordinates.

*Proof.* This proof is little different from the corresponding one in Ricci soliton case. We only sketch the frame of argument and one may refer to the proof of Lemma 4.3 in [17] for details.

We let  $D^1$  be the 2-dimensional distribution spanned by  $E_1 = \nabla s$  and  $E_2$ . And let  $D^2$  be the one spanned by  $E_3$  and  $E_4$ . Then  $D^1$  and  $D^2$  are both integrable by Lemma 5.1. We may consider the coordinates  $(x_1, x_2, x_3, x_4)$  from Lemma 4.2 of [17], so that  $D^1$  is tangent to the 2-dimensional level sets  $\{(x_1, x_2, x_3, x_4) \mid x_3, x_4 \text{ constants}\}$  and  $D^2$  is tangent to the level sets  $\{(x_1, x_2, x_3, x_4) \mid x_1, x_2 \text{ constants}\}$ . As  $D_1$  and  $D_2$  are orthogonal, we get the metric description for  $g$  as follows;

$g = g_{11}dx_1^2 + g_{12}dx_1 \odot dx_2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{34}dx_3 \odot dx_4 + g_{44}dx_4^2$ , where  $\odot$  is the symmetric tensor product and  $g_{ij}$  are functions of  $(x_1, x_2, x_3, x_4)$ .

Using Lemma 2.4, Lemma 3.1 and Lemma 5.1, one then shows that

$$g_{11}dx_1^2 + g_{12}dx_1 \odot dx_2 + g_{22}dx_2^2 = ds^2 + p(s)^2 dt^2,$$

and

$$g_{33}dx_3^2 + g_{34}dx_3 \odot dx_4 + g_{44}dx_4^2 = h(s)^2 \tilde{g},$$

for some functions  $p = p(s)$  and  $h = h(s)$  and a 2-dimensional metric  $\tilde{g}$ .

One can use Lemma 5.1 of [17] to prove that  $\tilde{g}$  has constant curvature, say  $k$ .  $\square$

## 6. ANALYSIS OF THE METRIC WHEN $\lambda_2 \neq \lambda_3 = \lambda_4$

We continue to suppose that  $\lambda_2 \neq \lambda_3 = \lambda_4$  for an adapted frame fields  $E_j$ ,  $j = 1, 2, 3, 4$ .

The metric  $\tilde{g}$  in (5.27) can be written locally;  $\tilde{g} = dr^2 + u(r)^2 d\theta^2$  on a domain in  $\mathbb{R}^2$  with polar coordinates  $(r, \theta)$ , where  $u''(r) = -ku$ . We set an orthonormal basis  $e_3 = \frac{\partial}{\partial r}$  and  $e_4 = \frac{1}{u(r)} \frac{\partial}{\partial \theta}$ .

**Lemma 6.1.** *For the local metric  $g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}$  with harmonic curvature satisfying (1.1) with nonconstant  $f$ , obtained in Lemma 5.2, if we set  $E_1 = \frac{\partial}{\partial s}$ ,  $E_2 = \frac{1}{p(s)} \frac{\partial}{\partial t}$ ,  $E_3 = \frac{1}{h(s)} e_3$  and  $E_4 = \frac{1}{h(s)} e_4$ , where  $e_3$  and  $e_4$  are as in the above paragraph, then we have the following. Here  $R_{ij} = R(E_i, E_j)$  and  $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ .*

$$\begin{aligned}
\nabla_{E_1} E_i &= 0, \text{ for } i = 2, 3, 4, & \nabla_{E_2} E_2 &= -\zeta_2 E_1, & \nabla_{E_3} E_3 &= -\zeta_3 E_1, \\
\zeta_2 &= \frac{p'}{p}, & \zeta_3 &= \zeta_4 = \frac{h'}{h} \\
R_{1221} &= -\frac{p''}{p} = -\zeta_2' - \zeta_2^2, & R_{1ii1} &= -\zeta_i' - \zeta_i^2 = -\frac{h''}{h}, \text{ for } i = 3, 4. \\
R_{11} &= -\zeta_2' - \zeta_2^2 - 2\zeta_3' - 2\zeta_3^2 = -\frac{p''}{p} - 2\frac{h''}{h}. \\
R_{22} &= -\zeta_2' - \zeta_2^2 - 2\zeta_2\zeta_3 = -\frac{p''}{p} - 2\frac{p'}{p}\frac{h'}{h}. \\
R_{33} &= R_{44} = -\zeta_3' - \zeta_3^2 - \zeta_3\zeta_2 - (\zeta_3)^2 + \frac{k}{h^2} = -\frac{h''}{h} - \frac{p'}{p}\frac{h'}{h} - \frac{(h')^2}{h^2} + \frac{k}{h^2}.
\end{aligned}$$

*Proof.* From the proof of Lemma 5.1, we already have  $\nabla_{E_1} E_2 = 0$ ,  $\nabla_{E_3} E_4 = \beta_3 E_3$  and  $\nabla_{E_4} E_3 = -\beta_4 E_4$ .

As  $\langle \nabla_{E_1} E_3, E_2 \rangle = -\langle E_3, \nabla_{E_1} E_2 \rangle = 0$ , one can readily get  $\nabla_{E_1} E_3 = \rho E_4$  for some function  $\rho$  and  $\nabla_{E_1} E_4 = -\rho E_3$ . We get  $\rho = 0$  by computing directly (in coordinates)  $\nabla_{E_1} E_3 = \nabla_{\frac{\partial}{\partial s}} \frac{1}{h(s)} \frac{\partial}{\partial r} = 0$ .

From Lemma 3.1 and Lemma 2.4 (iii);  $(\lambda_2 - \lambda_i) \langle \nabla_{E_2} E_2, E_i \rangle = E_i(\lambda_2) = 0$  for  $i = 3, 4$  and  $\langle \nabla_{E_2} E_2, E_1 \rangle = -\langle E_2, \nabla_{E_2} E_1 \rangle = -\zeta_2(s)$ . So,  $\nabla_{E_2} E_2 = -\zeta_2(s) E_1$ . By similar argument,  $\nabla_{E_3} E_3 = -\zeta_3 E_1 - \beta_3 E_4$ ,  $\nabla_{E_4} E_4 = -\zeta_4 E_1 + \beta_4 E_3$ , for some functions  $\beta_3$  and  $\beta_4$ . Direct coordinates computation gives  $\beta_3 = 0$ .

Then  $\nabla_{E_2} E_3 = q E_4$  for some function  $q$  and  $\nabla_{E_2} E_4 = -q E_3$ . One computes directly  $q = 0$ . We simply get  $\nabla_{E_3} E_2 = 0$  and  $\nabla_{E_4} E_2 = 0$ .

We compute directly that  $\nabla_{E_2} E_1 = \frac{p'}{p} E_2$  and  $\nabla_{E_3} E_1 = \frac{h'}{h} E_3$  so that (3.12) gives  $\zeta_2 = \frac{p'}{p}$  and  $\zeta_3 = \zeta_4 = \frac{h'}{h}$ . We can also get  $\nabla_{E_3} E_4 = 0$ ,  $\nabla_{E_4} E_3 = -\beta_4 E_4$ , where  $\beta_4 = \frac{u'(r)}{h(s)u(r)}$ .

These computations would help to compute the curvature components.  $\square$

We denote  $a := \zeta_2$  and  $b := \zeta_3$ .

**Lemma 6.2.** *For the local metric  $g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}$  with harmonic curvature satisfying (1.1) with nonconstant  $f$ , obtained in Lemma 5.2, it holds that*

$$(6.28) \quad \left(ab + \frac{R}{12}\right)b = 0.$$

*Proof.* (4.16) gives

$$(6.29) \quad 2a' + 2a^2 + 2ab + \frac{R}{3} = 0,$$

$$(6.30) \quad 2b' + 3b^2 + ab - \frac{k}{h^2} + \frac{R}{3} = 0.$$



From  $\nabla df(E_i, E_i) = f(Rc - \frac{R}{3}g)(E_i, E_i) + xR(E_i, E_i) + y(R)$ , we get  $-(\nabla_{E_i} E_i)f = f(R_{ii} - \frac{R}{3}) + xR_{ii} + y(R) = -fR_{1ii1} + xR_{ii} + y(R)$ , for  $i = 2, 3$ . From Lemma 6.1 we have

$$(6.31) \quad f'a = f(a' + a^2) - x(a' + a^2 + 2ab) + y(R),$$

$$(6.32) \quad f'b = f(b' + b^2) - x(b' + 2b^2 + ab - \frac{k}{h^2}) + y(R).$$

From the harmonic curvature condition we have;

$$\begin{aligned} 0 &= \nabla_{E_1} R_{22} - \nabla_{E_2} R_{12} = \nabla_{E_1}(R_{22}) + R(\nabla_{E_2} E_1, E_2) + R(\nabla_{E_2} E_2, E_1) \\ &= (R_{22})' + a(R_{22} - R_{11}) \\ &= (-a' - a^2 - 2ab)' + a(-2ab + 2b' + 2b^2) \\ (6.33) \quad &= -a'' - 2aa' - 2a'b - 2a^2b + 2ab^2. \end{aligned}$$

Differentiate (6.29) to get  $a'' + 2aa' + a'b + ab' = 0$ . Together with (6.33) we obtain

$$(6.34) \quad ab' - a'b - 2a^2b + 2ab^2 = 0.$$

Put (6.29) and (6.30) into (6.34) to get;

$$\begin{aligned} 0 &= -a(3b^2 + ab - \frac{k}{h^2} + \frac{R}{3}) + 2(a^2 + ab + \frac{R}{6})b - 4a^2b + 4ab^2 \\ &= a\frac{k}{h^2} + \frac{R}{3}(b - a) + 3ab(b - a). \end{aligned}$$

Then, as  $a \neq b$ ,

$$(6.35) \quad \frac{a}{a-b} \frac{k}{h^2} = \frac{R}{3} + 3ab.$$

From (6.31) and (6.32) we get

$$\frac{f'}{f}(a-b) = (a' + a^2 - b' - b^2) - \frac{x}{f}(a' + a^2 + 2ab - b' - 2b^2 - ab + \frac{k}{h^2}).$$

With (6.30) and (6.29), the above gives

$$2\frac{f'}{f}(a-b) = (1 + \frac{x}{f})(b^2 - ab - \frac{k}{h^2}).$$

Then by (6.35),  $2\frac{f'}{f}a = (1 + \frac{x}{f})(-ab - \frac{ka}{h^2(a-b)}) = (1 + \frac{x}{f})(-4ab - \frac{R}{3})$ .

Meanwhile, (6.31) and (6.29) gives  $f'a = -f(ab + \frac{R}{6}) - x(ab - \frac{R}{6}) + y(R)$ , so

$$2\frac{f'}{f}a = -2(ab + \frac{R}{6}) - \frac{2x}{f}(ab - \frac{R}{6}) + \frac{2y(R)}{f}, \text{ which should equal } (1 + \frac{x}{f})(-4ab - \frac{R}{3}).$$

So we obtain

$$(6.36) \quad x(ab + \frac{R}{3}) + y(R) = -fab.$$

Differentiating (6.36) and dividing by  $f$ ,

$$\frac{f'}{f}ab = -\frac{x}{f}(a'b + ab') - (a'b + ab').$$

From (6.31) we get  $\frac{f'}{f}ab = (a' + a^2)b - \frac{x}{f}(a' + a^2 + 2ab)b + \frac{yb}{f}$ ,

Equating the above two and arranging terms, we get

$\frac{x}{f}(-ab' + a^2b + 2ab^2) = 2a'b + ab' + a^2b + \frac{yb}{f}$ . Use (6.36) to get

$$(6.37) \quad \frac{x}{f}(-ab' + a^2b + 3ab^2 + \frac{R}{3}b) = 2a'b + ab' + a^2b - ab^2.$$

Using (6.34) and (6.29), the left hand side of (6.37) equals  $\frac{x}{f}(6ab^2 + \frac{R}{2}b)$ , while the right hand side equals  $-(6ab^2 + \frac{R}{2}b)$ .

So we get  $(1 + \frac{x}{f})(6ab + \frac{R}{2})b = 0$ . Then  $(ab + \frac{R}{12})b = 0$ .

□

**Proposition 6.3.** *For the local metric  $g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}$  with harmonic curvature satisfying (1.1) with nonconstant  $f$ , obtained in Lemma 5.2, suppose that  $ab = -\frac{R}{12}$ .*

*Then  $R = 0$ ,  $y(0) = 0$  and  $p$  is a constant. The metric  $g$  is locally isometric to a domain in the non-conformally-flat static space  $(W^3 \times \mathbb{R}^1, g_W + dt^2)$  of Example 3 in Subsection 2.1. And  $f = ch'(s) - x$ .*

*Proof.* As  $ab = -\frac{R}{12}$ , (6.36) gives  $\frac{R}{4}x + y(R) = \frac{R}{12}f$ .

If  $R \neq 0$ , then  $f$  is a constant, a contradiction to the hypothesis. Therefore  $R = 0$ . Then  $y(0) = 0$  from (6.36). From (6.29),  $a' + a^2 = 0$  and we have two cases: (i)  $a = \frac{1}{s+c}$  for a constant  $c$  or (ii)  $a = 0$ .

Case (i);  $a = \frac{1}{s+c}$ .

From (6.31),  $f'a = 0$ , so  $f$  is a constant, a contradiction to the hypothesis.

Case (ii);  $a = 0$ , i.e.  $p$  is a constant.

From (6.32) and (6.30), we get  $f'\frac{h'}{h} = (f+x)\frac{h''}{h}$ . If  $h'$  vanishes, we get  $\lambda_2 = \lambda_3$  a contradiction. So we may assume that  $h$  is not constant. Then  $ch' = f+x$  for a constant  $c \neq 0$ . Evaluating (1.1) at  $(E_1, E_1)$ ,

$$(6.38) \quad f'' = (f+x)R(E_1, E_1) - \frac{R}{3}f + y(R),$$

Here we get  $f'' = -2(f+x)\frac{h''}{h}$ , so  $h''' = -2h'\frac{h''}{h}$ . Hence, for a constant  $\alpha$ ,

$$(6.39) \quad h^2 h'' = \alpha.$$

From (6.30),  $0 = 2b' + 3b^2 - \frac{k}{h^2} = 2(\frac{h''}{h}) + (\frac{h'}{h})^2 - \frac{k}{h^2} = \frac{2\alpha}{h^3} + (\frac{h'}{h})^2 - \frac{k}{h^2}$ . So we have

$$(6.40) \quad (h')^2 + \frac{2\alpha}{h} - k = 0.$$

We have got exactly (2.10) and (2.11) in the case  $R = 0$  and  $n = 3$ . At this point we may write  $g = ds^2 + dt^2 + h(s)^2 \tilde{g} = (k - \frac{2\alpha}{h})^{-1} dh^2 + dt^2 + h(s)^2 \tilde{g}$ .

When  $\alpha = 0$ ,  $(h')^2 = k \geq 0$ . As  $h$  is not constant,  $k > 0$ . When  $h' = \pm\sqrt{k} \neq 0$ ,  $h = \pm\sqrt{k}s + c_0$ , for a constant  $c_0$ . One can see that  $g$  is a flat metric, a contradiction to  $\lambda_2 = \lambda_3$ .

When  $\alpha > 0$ , then  $k > 0$  from (6.40). We set  $r := \frac{h}{\sqrt{k}}$ , and then  $g = (1 - \frac{2\alpha}{k\sqrt{kr}})^{-1}dr^2 + dt^2 + r^2\tilde{g}_1$ , where  $\tilde{g}_1$  is the metric of constant curvature 1 on  $S^2$ . When  $\alpha < 0$ , the 3-d metric  $(1 - \frac{2\alpha}{k\sqrt{kr}})^{-1}dr^2 + r^2\tilde{g}_1$  corresponds to the case III.1 of Kobayashi's. It is incomplete as explained in his proposition 2.4.

In these two cases of  $\alpha > 0$  and  $\alpha < 0$ , we get the same Riemannian metrics as those of static spaces  $(W^3 \times \mathbb{R}^1, g_W + dt^2)$  explained in Example 3. And  $f = ch' - x$ .

Conversely, these metrics have harmonic curvature and satisfy (1.1) with the above  $f$ . Indeed, nontrivial components of (1.1) are (6.31), (6.32) and (6.38) whereas the harmonic curvature condition essentially consist of (6.33) and the equation  $\nabla_{E_1}R_{33} - \nabla_{E_3}R_{13} = 0$ ; all these can be verified from  $a = R = y(0) = 0$  and  $h, f$  which satisfy (6.39), (6.40) and  $f = ch' - x$ .  $\square$

**Proposition 6.4.** *For the local metric  $g = ds^2 + p(s)^2dt^2 + h(s)^2\tilde{g}$  with harmonic curvature satisfying (1.1) with nonconstant  $f$ , obtained in Lemma 5.2, suppose that  $b = 0$  and that  $ab = 0 \neq -\frac{R}{12}$ . Then the followings hold;*

(i)  $x\frac{R}{3} + y(R) = 0$ .

(ii) *when  $R > 0$ ,  $g$  is locally isometric to the Riemannian product  $\mathbb{S}^2(\frac{R}{6}) \times \mathbb{S}^2(\frac{R}{3})$  and  $f = c_1 \cos(\sqrt{\frac{R}{6}}s) - x$  for any constant  $c_1$ , where  $s$  is the distance on  $\mathbb{S}^2(\frac{R}{6})$  from a point.*

(iii) *when  $R < 0$ ,  $g$  is locally isometric to  $(\mathbb{H}^2(\frac{R}{6}) \times \mathbb{H}^2(\frac{R}{3}), g_{\frac{R}{6}} + g_{\frac{R}{3}})$ , where  $g_{\delta}$  is the 2-dimensional Riemannian metric of constant curvature  $\delta$ . The metric  $g_{\frac{R}{6}}$  can be written as  $g_{\frac{R}{6}} = ds^2 + p(s)^2dt^2$  where  $p'' + \frac{R}{6}p = 0$  and then  $f = c_2p' - x$  for any constant  $c_2$ .*

*Proof.* As  $b = 0$ , (6.36) gives (i). Next, (6.30) gives  $\frac{k}{h^2} = \frac{R}{3}$  and (6.29) gives  $a' + a^2 + \frac{R}{6} = \frac{p''}{p} + \frac{R}{6} = 0$ . These and (6.31) gives

$$(6.41) \quad f'a = -\frac{R}{6}(f + x).$$

Assume  $R > 0$ . Set  $r_0 = \sqrt{\frac{R}{6}}$ . For some constants  $C_1 \neq 0$  and  $s_0$ ,  $p = C_1 \sin(r_0(s + s_0))$  so that  $a = r_0 \cot(r_0(s + s_0))$ . Then (6.41) and (i)

gives  $f = c_1 \cos(r_0(s + s_0)) - x$ . Then  $g = ds^2 + \sin^2(r_0(s + s_0))dt^2 + \tilde{g}_{\frac{R}{3}}$  by absorbing a constant into  $dt^2$  and using  $\frac{k}{h^2} = \frac{R}{3}$ .

Replacing  $s + s_0$  by new  $s$ , we have  $g = ds^2 + \sin^2(r_0s)dt^2 + \tilde{g}_{\frac{R}{3}}$ . Here  $s$  becomes the distance on  $\mathbb{S}^2(\frac{R}{6})$  from a point. And  $f = c_1 \cos(r_0s) - x$ .

Assume  $R < 0$ . One can argue similarly as above and get  $g = ds^2 + (c_3e^{r_1s} + c_4e^{-r_1s})^2dt^2 + \tilde{g}_{\frac{R}{3}}$  and  $f = c_5e^{r_1s} + c_6e^{-r_1s} - x$ , where  $r_1 = \sqrt{-\frac{R}{6}}$  and  $c_3c_6 + c_4c_5 = 0$ .

Conversely, the above product metrics clearly have harmonic curvature. One can check they satisfy (1.1). Indeed, as in the proof of Proposition 6.3 one may check (6.31), (6.32), (6.38).  $\square$

## 7. LOCAL 4-DIMENSIONAL SPACE WITH HARMONIC CURVATURE

We first treat the remaining case of  $\lambda_2 = \lambda_3 = \lambda_4$  and then give the proof of Theorem 1.1.

**Proposition 7.1.** *Let  $(M, g, f)$  be a four dimensional Riemannian manifold with harmonic curvature satisfying (1.1) with non constant  $f$ . Suppose that  $\lambda_2 = \lambda_3 = \lambda_4 \neq \lambda_1$  for an adapted frame field in an open subset  $U$  of  $M_r \cap \{\nabla f \neq 0\}$ .*

*Then for each point  $p_0$  in  $U$ , there exists a neighborhood  $V$  of  $p_0$  in  $U$  where  $g$  is a warped product;*

$$(7.42) \quad g = ds^2 + h(s)^2 \tilde{g},$$

*where  $h$  is a positive function and the Riemannian metric  $\tilde{g}$  has constant curvature, say  $k$ . In particular,  $g$  is conformally flat.*

*As a Riemannian manifold,  $(M, g)$  is locally one of Kobayashi's warped product spaces, as described in the section 2 and 3 of [18] so that*

$$(7.43) \quad h'' + \frac{R}{12}h = ah^{-3}$$

*for a constant  $a$ , and integrating it for some constant  $k$ ,*

$$(7.44) \quad (h')^2 + \frac{2a}{n-2}h^{2-n} + \frac{R}{n(n-1)}h^2 = k.$$

*And  $f$  is a non-constant solution to;*

$$(7.45) \quad h'f' - fh'' = x(h'' + \frac{R}{3}h) + y(R)h.$$

*Conversely, any  $(h, f)$  satisfying (7.43), (7.44) and (7.45) gives rise to  $(g, f)$  which has harmonic curvature and satisfies (1.1).*

*Proof.* To prove that  $g$  is in the form of (7.42), we may use Lemma 2.3 (v) and Lemma 2.4 (iii), (iv). For actual proof we refer to that of Proposition 7.1 of [17] since the argument is almost the same as in the gradient Ricci soliton case. To prove that  $\tilde{g}$  has constant curvature, we use Derdziński's Lemma 4 in [10]. Then (7.42) metric is conformally flat.

In the setting of Lemma 2.3,  $f$  is a function of  $s$  only. For  $g = ds^2 + h(s)^2 \tilde{g}$ , in a local adated frame field, we have

$$(7.46) \quad \begin{aligned} R_{11} &= -3\frac{h''}{h}, & R_{ii} &= -\frac{h''}{h} - 2\frac{(h')^2}{h^2} + 2\frac{k}{h^2} \\ R_{ij} &= 0 \quad \text{for } i \neq j \\ R &= -6\frac{h''}{h} - 6\frac{(h')^2}{h^2} + 6\frac{k}{h^2} \end{aligned}$$

Feeding  $(E_i, E_i)$ ,  $i = 1, 2$  to (1.1) we may get;

$$(7.47) \quad f'' = -3f\frac{h''}{h} - f\frac{R}{3} - 3x\frac{h''}{h} + y(R),$$

$$(7.48) \quad h'f' - fh'' = x(h'' + \frac{R}{3}h) + y(R)h.$$

Differentiating (7.48) and using (7.47); we get  $(f+x)\{h''' + 3\frac{h''h'}{h} + \frac{R}{3}h'\} = 0$ . As  $f \neq -x$ , we get  $h''' + 3\frac{h''h'}{h} + \frac{R}{3}h' = 0$ . Multiply this by  $h^3$ , we get  $(h^3h'' + \frac{R}{12}h^4)' = 0$ . Then we have (7.43) and then (7.44). Kobayashi solved these completely according to each parameter and initial condition.

One can check that any  $h$  and  $f$  satisfying (7.48), (7.43) and (7.44) satisfy (7.46) and (7.47).  $\square$

We are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Recall that we have already discussed the case  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$  in Example 1 of Subsection 2.1.2. The conformally flat spaces in Example 1 belong to the type (iv) of Theorem 1.1; especially they have  $a = 0$  in (1.6) and (1.7).

As the metric  $g$  and  $f$  are real analytic, the Ricci-eigen values  $\lambda_i$ 's are real analytic on  $M_r \cap \{\nabla f \neq 0\}$ . And  $\zeta_i$ 's are real analytic from (3.12). So we can combine Proposition 4.2, Lemma 6.2, Proposition 6.3, 6.4, 7.1 and Example 1 of Subsection 2.1.2, to obtain a classification of 4-dimensional *local* spaces with harmonic curvature satisfying (1.1) as Theorem 1.1.  $\square$

**Remark 7.2.** In the statement of Theorem 1.1, among the types (i)~(iv), there is possibly only one type of neighborhoods  $V$  on a *connected* space  $(M, g, f)$ ; it holds from continuity argument of Riemannian metrics. Then one can prove that  $\tilde{M} = M$  if  $M$  is of type (i), (ii) or (iii).

## 8. COMPLETE 4-DIMENSIONAL SPACE WITH HARMONIC CURVATURE

It is not hard to describe complete spaces corresponding to (i),(ii), (iii) of Theorem 1.1.

For complete conformally flat case corresponding to (iv) of Theorem 1.1, we may use Theorem 3.1 of Kobayashi's classification [18]. Then  $(M, g)$  can be either  $\mathbb{S}^4$ ,  $\mathbb{H}^4$ , a flat space or one of the spaces in Example 1~5 in [18]. Now our task is to determine  $f$ , which is described by (1.8).

We first recall the spaces in Example 3~5 in [18]. Any space in Example 3 and 4 in [18] is a quotient of a warped product  $\mathbb{R} \times_h N(1)$  where  $h$  is a smooth periodic function on  $\mathbb{R}$ ; recall that  $N(k)$  is a Riemannian manifold of constant sectional curvature  $k$ . Any space in Example 5 in [18] is a quotient of a warped product  $\mathbb{R} \times_h N(k)$  where  $h$  is smooth on  $\mathbb{R}$ . Here  $h \geq \rho_1 > 0$ .

We verify the following lemma.

**Lemma 8.1.** *For any one of the spaces in Example 3, 4 and 5 in [18], the following holds.*

- (i) *The solution  $f$  to (1.1) can be defined and smooth on  $\mathbb{R}$ .*
- (ii) *If  $h$  is periodic and  $x\frac{R}{3} + y(R) = 0$ , then  $f$  is periodic.*

*Proof.* As stated in Theorem 7.1, any  $(h, f)$  satisfying (7.43), (7.44) and (7.45) gives rise to  $(g, f)$  which satisfies (1.1). So,  $(h, f)$  satisfies (7.47).

Choose some point  $s_0$  with  $h''(s_0) \neq 0$ . We consider the ODE for any constant  $c$ ,

$$(8.49) \quad f'' = -f\left(\frac{R}{12} + 3ah^{-4}\right) + 3x\left(\frac{R}{12} - ah^{-4}\right) + y(R),$$

with initial conditions  $f'(s_0) = c$  and  $f(s_0) = \frac{ch'(s_0) - \{x(h''(s_0) + \frac{R}{3}h(s_0)) + y(R)h(s_0)\}}{h''(s_0)}$  so that (1.8) holds at  $s_0$ . Note that (8.49) is equivalent to (7.47) since  $h$  satisfies (1.6).

As  $h$  exists smoothly on  $\mathbb{R}$  as a solution of (1.6), by global Lipschitz continuity of the right hand side of (8.49), the solution  $f$  exists globally on  $\mathbb{R}$ .

We get from (1.6);

$$(8.50) \quad h''' = -\left(\frac{R}{12} + 3ah^{-4}\right)h'.$$

Then by (8.49) and (8.50) it satisfies  $h'f'' - fh''' = x(h''' + \frac{R}{3}h') + y(R)h'$ , which is the derivative of (1.8). So, (1.8) holds on  $\mathbb{R}$ . As  $h$  and  $f$  satisfy (1.8), the induced  $(g, f)$  satisfies (1.1) on  $\mathbb{R}$ .

If  $x\frac{R}{3} + y(R) = 0$ , then from (1.8) we get  $f(s) = -x + Ch'(s)$  for a constant  $C$ , which is periodic as  $h$ .  $\square$

About Lemma 8.1 (ii), we note that if  $x\frac{R}{3} + y(R) \neq 0$  and  $h$  is periodic, then the periodicity of  $f$  should be checked by computation.

We are ready to state;

**Theorem 8.2.** *Let  $(M, g)$  be a four dimensional complete Riemannian manifold with harmonic curvature, satisfying (1.1) with non-constant  $f$ . Then it is one of the following;*

(8.2-i)  $(M, g)$  is isometric to a quotient of  $\mathbb{S}^2(\frac{R}{6}) \times \mathbb{S}^2(\frac{R}{3})$  with  $R > 0$ , where And  $f = c_1 \cos(\sqrt{\frac{R}{6}}s) - x$  for any constant  $c_1$ , where  $s$  is the distance on  $\mathbb{S}^2(\frac{R}{6})$  from a point. It holds that  $x\frac{R}{3} + y(R) = 0$ .

(8.2-ii)  $(M, g)$  is isometric to a quotient of  $(\mathbb{H}^2(\frac{R}{6}) \times \mathbb{H}^2(\frac{R}{3}), g_{\frac{R}{6}} + g_{\frac{R}{3}})$  with  $R < 0$ . And  $f = c_2 \cosh(s) - x$  for any constant  $c_2$ , where  $s$  is the distance function on  $\mathbb{H}^2(\frac{R}{6})$  from a point. It holds that  $x\frac{R}{3} + y(R) = 0$ .

(8.2-iii)  $(M, g)$  is isometric to a quotient of one of the static spaces in Example 3 of Subsection 2.1.2, which is the Riemannian product  $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2\tilde{g})$  of  $\mathbb{R}^1$  and some 3-dimensional conformally flat static space  $(W^3 = \mathbb{R}^1 \times \mathbb{S}^2(1), ds^2 + h(s)^2\tilde{g})$  with zero scalar curvature, which contains the space section of the Schwarzschild space-time

And  $f = c \cdot h'(s) - x$  for a constant  $c$ . It holds that  $R = y(0) = 0$ .

(8.2-iv)  $(M, g)$  is conformally flat. It is either  $\mathbb{S}^4, \mathbb{H}^4$ , a flat space or one of the spaces in Example 1~5 in [18]. Below we describe  $f$  in each subcase.

(8.2-iv-1)  $\mathbb{S}^4(k^2)$  with the metric  $g = ds^2 + \frac{\sin(ks)^2}{k^2}g_1$ , for any constant  $c$ ,

$$f(s) = c \cdot \cos(ks) + 3x + \frac{y(12k^2)}{k^2}.$$

(8.2-iv-2)  $\mathbb{H}^4(-k^2)$ , with  $g = ds^2 + \frac{\sinh(ks)^2}{k^2}g_1$ , for any constant  $c$ ,

$$f(s) = c \cdot \cosh(ks) + 3x - \frac{y(-12k^2)}{k^2}.$$

(8.2-iv-3) A flat space,  $f = a + \sum_i b_i x_i + \frac{y(0)}{2} x_i^2$  in a local Euclidean coordinates  $x_i$ , for constants  $a$  and  $b_i$ .

(8.2-iv-4) *Example 1 and 2 in [18]; the Riemannian product  $(\mathbb{R} \times N(k), ds^2 + g_k)$  or its quotient,  $k \neq 0$ , where  $N(k)$  is 3-dimensional complete space of constant sectional curvature  $k$ ,*

$$f = c_1 \sin \sqrt{\frac{R}{3}}s + c_2 \cos \sqrt{\frac{R}{3}}s - x \text{ when } R > 0, \text{ or}$$

$$f = c_1 \sinh \sqrt{-\frac{R}{3}}s + c_2 \cosh \sqrt{-\frac{R}{3}}s - x \text{ when } R < 0.$$

*It holds that  $x\frac{R}{3} + y(R) = 0$  and  $R = 6k$ .*

(8.2-iv-5) *Example 3 and 4 in [18]; a warped product  $\mathbb{R} \times_h N(1)$  or its quotient, where  $h$  is a periodic function on  $\mathbb{R}$ ,  $f$  is on  $\mathbb{R}$ , satisfying (1.8).*

(8.2-iv-6) *Example 5 in [18]; a warped product  $\mathbb{R} \times_h N(k)$  where  $h$  is defined on  $\mathbb{R}$ ,  $f$  is on  $\mathbb{R}$ , satisfying (1.8).*

*Proof.* To obtain (8.2-i), (8.2-ii) and (8.2-iii), we use continuity argument of Riemannian metrics from Theorem 1.1. To describe  $f$  in the subcases of (8.2-iv), we use (1.8) and (7.47).  $\square$

#### 9. 4-D STATIC SPACES WITH HARMONIC CURVATURE

In this section we study static spaces, i.e. those satisfying (1.2). As explained in the introduction, to study local static spaces is interesting due to Corvino's local deformation theory of scalar curvature. Here we state local classification which can be read off from Theorem 1.1;

**Theorem 9.1.** *Let  $(M, g, f)$  be a four dimensional (not necessarily complete) static space with harmonic curvature and non-constant  $f$ . Then for each point  $p$  in some open dense subset  $\tilde{M}$  of  $M$ , there exists a neighborhood  $V$  of  $p$  with one of the following properties;*

(9.1-i)  *$(V, g)$  is isometric to a domain in  $\mathbb{S}^2(\frac{R}{6}) \times \mathbb{S}^2(\frac{R}{3})$  with  $R > 0$ . And  $f = c_1 \cos(\sqrt{\frac{R}{6}}(s + s_0))$ , where  $s$  is the distance function on  $\mathbb{S}^2(\frac{R}{6})$  from a point and  $c_1, s_0$  are constants.*

(9.1-ii)  *$(V, g)$  is isometric to a domain in  $(\mathbb{H}^2(\frac{R}{6}) \times \mathbb{H}^2(\frac{R}{3}), g_{\frac{R}{6}} + g_{\frac{R}{3}})$  with  $R < 0$ . If we express  $g_{\frac{R}{6}}$  as  $g_{\frac{R}{6}} = ds^2 + p(s)^2 dt^2$  with  $p'' + \frac{R}{6}p = 0$ , then  $f = c_2 p'$  for any constant  $c_2$ .*

(9.1-iii)  *$(V, g)$  is isometric to a domain in one of the static spaces in Example 3 of Subsection 2.1.2, which is the Riemannian product  $\mathbb{R}^1 \times W^3$  of  $\mathbb{R}^1$  and some 3-dimensional conformally flat static space  $(W^3, ds^2 + h(s)^2 \tilde{g})$  with zero scalar curvature. And  $f = ch'$ .*

(9.1-iv)  *$(V, g)$  is conformally flat. So, it is one of the warped product metrics of the form  $ds^2 + h(s)^2 g_k$  whose existence is described in the section 2 of [18]. The function  $h$  satisfies (1.6) and (1.7), and we have  $f(s) = Ch'(s)$ .*



For complete conformally flat case corresponding to (9.1-iv) in Theorem 9.1, if we use Theorem 3.1 of Kobayashi's classification, we get either  $\mathbb{S}^4$ ,  $\mathbb{H}^4$ , a flat space or one of the spaces in Example 1~5 in [18]. We may obtain classification of complete four dimensional static spaces with harmonic curvature;

**Theorem 9.2.** *Let  $(M, g, f)$  be a complete four dimensional static space with harmonic curvature. Then it is one of the following;*

(9.2-i)  $(M, g)$  is isometric to a quotient of  $\mathbb{S}^2(\frac{R}{6}) \times \mathbb{S}^2(\frac{R}{3})$  with  $R > 0$ . And  $f = c_1 \cos(\sqrt{\frac{R}{6}}s)$ , where  $s$  is the distance function on  $\mathbb{S}^2(\frac{R}{6})$  from a point.

(9.2-ii)  $(M, g)$  is isometric to a quotient of  $\mathbb{H}^2(\frac{R}{6}) \times \mathbb{H}^2(\frac{R}{3})$  with  $R < 0$ . And

$f = c_2 \cosh(\sqrt{\frac{-R}{6}}s)$ , where  $s$  is the distance function on  $\mathbb{H}^2(\frac{R}{6})$  from a point.

(9.2-iii)  $(M, g)$  is isometric to a quotient of the Riemannian product  $(\mathbb{R}^1 \times W^3, dt^2 + \tilde{g})$ , where  $(W^3, \tilde{g})$  denotes the warped product manifold on the smooth product  $\mathbb{R}^1 \times \mathbb{S}^2(1)$  which contains the space section of the Schwarzschild space-time; see Example 3 of Subsection 2.1.2.

(9.2-iv)  $(M, g, f)$  is  $\mathbb{S}^4$ ,  $\mathbb{H}^4$ , a flat space or one of the spaces in Example 1~5 in [17].

(9.2-v)  $g$  is a complete Ricci-flat metric with  $f$  a constant function.

*Proof.* It follows from Theorem 8.2. When  $f$  is nonzero constant,  $g$  is clearly Ricci-flat. So we get (v).  $\square$

Fischer-Marsden [12] made the conjecture that any closed static space is Einstein. But it was disproved by conformally flat examples in [14, 18]. Now we ask

**Question 1:** Does there exist a closed static space which does not have harmonic curvature?

The space in (9.2-iii) of Theorem 9.2 has three distinct Ricci-eigenvalues. We only know examples of static spaces with at most three distinct Ricci-eigenvalues. So, we ask;

**Question 2:** Does there exist a static space with more than three distinct Ricci-eigenvalues? Is there a limit on the number of distinct Ricci-eigenvalues for a static space?

## 10. MIAO-TAM CRITICAL METRICS AND $V$ -CRITICAL SPACES

In this section we treat Miao-Tam critical metrics. These metrics originate from [15] where Miao and Tam studied the critical points of the volume

functional on the space  $\mathcal{M}_\gamma^K$  of metrics with constant scalar curvature  $K$  on a compact manifold  $M$  with a prescribed metric  $\gamma$  at the boundary of  $M$ . Miao-Tam critical metrics are precisely described [16] in case they are Einstein or conformally flat.

Here we first describe 4-d metrics with harmonic curvature which have a nonzero solution  $f$  to (1.3). We do not assume the condition  $f|_\Sigma = 0$  but still can show that any such metric must be conformally flat;

**Theorem 10.1.** *Let  $(M, g)$  be a four dimensional (not necessarily complete) Riemannian manifold with harmonic curvature, satisfying (1.3) with non-constant  $f$ . Then  $(M, g)$  is conformally flat. It is one of the warped product metrics of the form  $ds^2 + h(s)^2 g_k$  whose existence is described in the section 2 of [18]. The function  $h$  satisfies (1.6) and (1.7), and  $f$  satisfies  $h' f' - f h'' = -\frac{h}{n-1}$ .*

*Proof.* The proof is immediate from Theorem 1.1; the cases (i)-(ii) of Theorem 1.1 require  $x \frac{R}{3} + y(R) = 0$  and (iii) requires  $y(0) = 0$ , which contradict to the condition  $x = 0$  and  $y(R) = -\frac{1}{3}$  that (1.3) has. The description of Theorem 1.1 (iv) holds for  $g$  and  $f$  of Theorem 10.1, and in particular  $g$  is conformally flat.  $\square$

Theorem 10.1 shows an advantage of our local approach over [1] in analyzing (1.3). In fact, the integration argument of [1, Lemma 5] only works for compact manifolds, but our analysis can resolve local solutions.

From Theorem 9.1 and 10.1 we can classify local 4-d  $V$ -static spaces with harmonic curvature;

**Theorem 10.2.** *Let  $(M, g, f)$  be a four dimensional (not necessarily complete)  $V$ -static space with harmonic curvature and non-constant  $f$ . Then for each point  $p$  in some open dense subset  $\tilde{M}$  of  $M$ , there exists a neighborhood  $V$  of  $p$  with one of the following properties;*

(10.2-i)  $(V, g)$  is isometric to a domain in  $\mathbb{S}^2(\frac{R}{6}) \times \mathbb{S}^2(\frac{R}{3})$  with  $R > 0$ . And  $f = c_1 \cos(\sqrt{\frac{R}{6}}(s + s_0))$ , where  $s$  is the distance function on  $\mathbb{S}^2(\frac{R}{6})$  from a point and  $c_1, s_0$  are constants.

(10.2-ii)  $(V, g)$  is isometric to a domain in  $(\mathbb{H}^2(\frac{R}{6}) \times \mathbb{H}^2(\frac{R}{3}), g_{\frac{R}{6}} + g_{\frac{R}{3}})$  with  $R < 0$ . If we express  $g_{\frac{R}{6}}$  as  $g_{\frac{R}{6}} = ds^2 + p(s)^2 dt^2$  with  $p'' + \frac{R}{6}p = 0$ , then  $f = c_2 p'$  for any constant  $c_2$ .

(10.2-iii)  $(V, g)$  is isometric to a domain in one of the static spaces in Example 3 of Subsection 2.1.2, which is the Riemannian product  $\mathbb{R}^1 \times W^3$  of  $\mathbb{R}^1$  and some 3-dimensional conformally flat static space  $(W^3, ds^2 + h(s)^2 \tilde{g})$  with zero scalar curvature. And  $f = ch'$  for any constant  $c$ .

(10.2-iv)  $(V, g)$  is conformally flat. It is one of the warped product metrics of the form  $ds^2 + h(s)^2 g_k$  whose existence is described in the section 2 of [18]. The function  $h$  satisfies (1.6) and (1.7), and we have  $f(s) = ch'(s)$  for any constant  $c$ .

(10.2-v)  $(V, g)$  is conformally flat. It is one of the warped product metrics of the form  $ds^2 + h(s)^2 g_k$  whose existence is described in the section 2 of [18]. The function  $h$  satisfies (1.6) and (1.7) and  $f$  is any constant multiple of a solution  $f_0$  satisfying  $h'f_0' - f_0h'' = -\frac{h}{n-1}$ .

Note that the last equation in (10.2-v) comes from (1.4), which allows any constant multiple of one solution.

As a Corollary of Theorem 10.1, we could state an extension of Miao-Tam's theorem 1.2 in [16] to the case of harmonic curvature. Instead we choose to state the following version, which is a twin to the corollary 1 of [1].

**Theorem 10.3.** *If  $(M^4, g, f)$  is a simply connected, compact Miao-Tam critical metric of harmonic curvature with boundary isometric to a standard sphere  $S^3$ . Then  $(M^4, g)$  is isometric to a geodesic ball in a simply connected space form  $\mathbb{R}^4, \mathbb{H}^4$  or  $\mathbb{S}^4$ .*

One can also make classification statements of complete spaces with harmonic curvature satisfying (1.3) or (1.4). We omit them.

Theorem 10.1 gives a speculation that it might hold in general dimension. So, we ask;

**Question 3:** Let  $(M, g)$  be an  $n$ -dimensional Miao-Tam critical metric with harmonic curvature. Is it conformally flat?

It is also interesting to find examples of non-conformally flat Miao-Tam critical metric in any dimension.

## 11. ON CRITICAL POINT METRICS

In this section we study a critical point metric, i.e. a Riemannian metric  $g$  on a manifold  $M$  which admits a non-zero solution  $f$  to (1.5). According to [13], these critical point metrics with harmonic curvature on closed manifolds in any dimension are Einstein.

On a closed manifold, by taking the trace of this equation,  $R$  must be positive and  $f$  satisfies  $\int_M f \, dv = 0$ . Here  $M$  is not necessarily closed and  $g$  may have non-positive scalar curvature. From Theorem 1.1, we can easily obtain the next theorem.

**Theorem 11.1.** *Let  $(M, g)$  be a four dimensional (not necessarily complete) Riemannian manifold with harmonic curvature, satisfying (1.5) with non-constant  $f$ . Then one of the following holds.*

(11.1-i)  $(M, g)$  is locally isometric to a domain in one of the static spaces of Example 3 in Subsection 2.1.2 below, which is the Riemannian product  $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2 \tilde{g})$  of  $\mathbb{R}^1$  and a 3-dimensional conformally flat static space  $(W^3, ds^2 + h(s)^2 \tilde{g})$  with zero scalar curvature. And  $f = c \cdot h'(s) - 1$ .

(11.1-ii)  $(M, g)$  is conformally flat and is locally one of the metrics whose existence is described in the section 2 of [18];  $g = ds^2 + h(s)^2 g_k$  where  $h$  and  $f$  satisfy (1.6), (1.7) and (1.8).

*Proof.* We have  $x \frac{R}{3} + y(R) = 0$  and  $R \neq 0$  in the cases (i), (ii) of Theorem 1.1. This is not compatible with (1.5).  $\square$

Complete spaces with harmonic curvature which admit a solution  $f$  to (1.5) are described in the next theorem. We obtain non-conformally-flat examples with zero scalar curvature in (11.2-i), which is in contrast to the above result of [13] for closed manifolds. The case (11.2-v) is also noteworthy; it is conformally flat with positive scalar curvature and the metric  $g$  can exist on a compact quotient but the function  $f$  can survive on the universal cover  $\mathbb{R} \times_h N(1)$ .

**Theorem 11.2.** *Let  $(M, g)$  be a four dimensional complete Riemannian manifold with harmonic curvature, satisfying (1.5) with non-constant  $f$ . Then  $(M, g)$  is one of the following;*

(11.2-i)  $(M, g)$  is isometric to a quotient of one of the static spaces of Example 3 in Subsection 2.1.2 below, which is the Riemannian product  $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2 \tilde{g})$  of  $\mathbb{R}^1$  and a 3-dimensional conformally flat static space  $(W^3, ds^2 + h(s)^2 \tilde{g})$  with zero scalar curvature which contains the space section of the Schwarzschild space-time. And  $f = c \cdot h'(s) - 1$  for a constant  $c$ .

(11.2-ii)  $\mathbb{S}^4(k^2)$  with the metric  $g = ds^2 + \frac{\sin^2(ks)}{k^2} g_1$ , with  $f(s) = c \cdot \cos(ks)$ .

(11.2-iii)  $\mathbb{H}^4(-k^2)$ , with  $g = ds^2 + \frac{\sinh(ks)^2}{k^2} g_1$ , with  $f(s) = c \cdot \cosh(ks)$ .

(11.2-iv) A flat space,  $f = a + \sum_i b_i x_i$  in a local Euclidean coordinates  $x_i$  and constants  $a, b_i$ .

(11.2-v) Example 3 in [18]; A warped product  $\mathbb{R} \times_h N(1)$  where  $h$  is a periodic function on  $\mathbb{R}$ ,  $f$  is smooth on  $\mathbb{R}$  but is not periodic. Here  $R > 0$ .

(11.2-vi) Example 5 in [18]; A warped product  $\mathbb{R} \times_h N(k)$  where  $h$  is defined on  $\mathbb{R}$ ,  $f$  is smooth on  $\mathbb{R}$ . Here  $R \leq 0$ .

*Proof.* We may check the list in Theorem 8.2. The spaces of (8.2-i) and (8.2-ii) in Theorem 8.2 are excluded as in the proof of Theorem 11.1. The space

for (8.2-iv-4) of Theorem 8.2, where  $R \neq 0$ , dose not satisfy the equation  $h' f' - f h'' = x(h'' + \frac{R}{3}h) + y(R)h$ ; when  $x = 1$ ,  $y(R) = -\frac{R}{4}$  and  $h = 1$ , it reduces to  $0 = \frac{R}{12}$ .

On the space of (8.2-iv-5) in Theorem 8.2,  $f$  is defined and smooth on  $\mathbb{R}$  by Lemma 8.1 (i). As  $x\frac{R}{3} + y(R) \neq 0$ , Lemma 8.1 (ii) does not apply. According to the section E.2 of [14],  $f$  cannot be periodic. This yields (11.2-v).  $\square$

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